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SG-PSEUDODIFFERENTIAL OPERATORS AND GELFAND-SHILOV SPACES

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1. Introduction. Linear partial differential operators, or more generally pseudodifferential operators, of SG-type (symbol global-type) are defined in $\mathbf{R}^n$ by imposing suitable algebraic asymptotics as $x \to \infty$ for the symbols of the operators. Basic examples are $-\Delta + 1$ and for $k \geq 1$

$$H = (1 + |x|^{2k})(-\Delta + 1) + L_1(x, D) \tag{0.1}$$

where $L_1(x, D)$ is a first order operator with polynomial coefficients of degree $2k - 1$. Let us refer to Parenti [**16**], Cordes [**5**], Schrohe [**21**], Egorov and Schulze [**10**], Schulze [**22**] for a precise definition and the corresponding pseudodifferential calculus in the frame of the Schwartz spaces $\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n)$. As an application for the SG-elliptic operators P, having H in (0.1) as prototype, the above mentioned authors construct parametrices and deduce in particular the following result of global regularity: all the solutions $u \in \mathcal{S}'(\mathbf{R}^n)$ of $Pu = f \in \mathcal{S}(\mathbf{R}^n)$ are of class $\mathcal{S}(\mathbf{R}^n)$. In particular, when P is self-adjoint, as we have for H in (0.1) if $L_1(x, D)$ is suitably chosen, there exists a system of eigenfunctions in the space $\mathcal{S}(\mathbf{R}^n)$; see, for example, Maniccia and Panarese [**15**] for the corresponding eigenvalue asymptotics.

Our aim in this paper is to obtain more precise information concerning the behavior for $x \to \infty$ and the local regularity of the solutions under related assumptions on the regularity of the coefficients. The functional frame, providing the two results simultaneously, is given here by the classes $S_\theta^\theta(\mathbf{R}^n), S_\theta^{\theta'}(\mathbf{R}^n), \theta > 1$, introduced by Gelfand and Shilov [**11**]. Let us recall that $S_\theta^\theta(\mathbf{R}^n)$ is a subclass of $\mathcal{S}(\mathbf{R}^n)$, combining the exponential decay $e^{-L|x|^{1/\theta}}$, $L > 0$, with the local Gevrey estimates of order θ, i.e., $|D^\alpha u(x)| \leq C^{|\alpha|+1}(\alpha!)^\theta$. In turn, the ultradistribution space $S_\theta^{\theta'}(\mathbf{R}^n)$ contains $\mathcal{S}'(\mathbf{R}^n)$ and admits as examples functions with growth $e^{L|x|^\sigma}$, $\sigma < 1/\theta$, see Section 1 for details. Observe that the

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