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ON BASIC EMBEDDINGS INTO THE PLANE

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ABSTRACT. A subset $K \subset \mathbf{R}^2$ is said to be *basic* if for each function $f\colon K \to \mathbf{R}$ there exist functions $g, h\colon \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$. If all the three functions in this definition are assumed to be *continuous* (*differentiable*), then the embedding is C^0*-basic* (C^1*-basic*). This notion appeared in studies of Hilbert's 13th problem on superpositions. We prove that *if a finite graph is C^0-basically embeddable in the plane, then it is C^1-basically embeddable in the plane.* In our proof we construct an explicit C^1-basic embedding and use the Skopenkov characterization of graphs C^0-basically embeddable in the plane. Our result is nontrivial because the plane contains graphs which are C^0-basic but not C^1-basic and graphs which are C^1-basic but not C^0-basic (Baran-Skopenkov). We also prove that *given any integer $k \geq 0$, there is a subset of the plane which is C^r-basic for each $0 \leq r \leq k$ but not C^r-basic for each $k < r \leq \omega$.*

1. Introduction. The notion of a basic embedding appeared implicitly in the Kolmogorov-Arnold solution of Hilbert's 13th problem [**1, 5, 6**]. A compactum $K \subset \mathbf{R}^2$ is said to be *basic* if, for each continuous function $f\colon K \to \mathbf{R}$ there exist continuous functions $g, h\colon \mathbf{R} \to \mathbf{R}$ such that $f(x,y) = g(x) + h(y)$ for each point $(x,y) \in K$. One can replace in the definition of a basic embedding *continuous* functions by *smooth* functions (by Lipschitz, Hölder, analytic, etc., functions) and obtain a notion of basic embeddability in a smooth, Lipschitz, Hölder, analytic, etc. sense.

This note is motivated by the following problems.

Problem 1. *Find conditions on a compactum $K \subset \mathbf{R}^2$, under which K is basically embeddable into the plane in the smooth sense.*

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