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HERON TRIANGLES VIA ELLIPTIC CURVES

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ABSTRACT. Given a positive integer n, one may ask if there is a right triangle with rational sides having area n. Such integers are called congruent numbers, and are closely related to elliptic curves of the form $y^2 = x^3 - n^2 x$. In this paper, we generalize this idea and show that there is a correspondence between positive integers n associated with arbitrary triangles with rational sides having area n and the family of elliptic curves $y^2 = x(x - n\,\tau)(x + n\,\tau^{-1})$ for nonzero rational τ.

1. Introduction. The Indian mathematician Brahmagupta, 598–668 A.D., considered triangles with integral sides and integral area. He showed that if such a triangle has sides of length a, b and c and has area n, then there are positive integers p, q and r such that

$$\begin{aligned} a &= q(p^2 + r^2) \\ b &= p(q^2 + r^2) \\ c &= (p + q)(p\,q - r^2) \end{aligned} \tag{1.1}$$

and

$$n = p\,q\,r(p + q)(p\,q - r^2);$$

as long as $p\,q > r^2$. (A modern proof can be found in [**4**].) In general, the sides and area are related by a formula first proved by Greek mathematician Heron of Alexandria (c. 10 A.D.–c. 75 A.D.):

$$n = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where} \quad s = \frac{a+b+c}{2}. \tag{1.2}$$

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