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CONGRUENCES AND RATIONAL EXPONENTIAL SUMS WITH THE EULER FUNCTION

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ABSTRACT. We give upper bounds for the number of solutions to congruences with the Euler function $\varphi(n)$ modulo an integer $q \geq 2$. We also give nontrivial bounds for rational exponential sums with $\varphi(n)/q$.

1. Introduction. Let $\varphi(n)$ denote the Euler function:

$$\varphi(n) = \#\{1 \leq a \leq n \mid \gcd(a,n) = 1\}.$$

For any integer $q \geq 2$, let $\mathbf{e}_q(z)$ denote the exponential function $\exp(2\pi i z/q)$, which is defined for all $z \in \mathbf{R}$.

In this paper, we give upper bounds for rational exponential sums of the form

$$S_a(x,q) = \sum_{n \leq x} \mathbf{e}_q(a\varphi(n)),$$

where $\gcd(a,q) = 1$, and x is sufficiently large. Our results are nontrivial for a wide range of values for the parameter q. In the special case where $q = p$ is a prime number, however, stronger results have been obtained in [**1**].

One of the crucial ingredients of [**1**] is an upper bound on the number solutions of a congruence with the Euler function. To be more precise, let $T(x,q)$ denote the number of positive integers $n \leq x$ such that $\varphi(n) \equiv 0 \pmod q$. The results of [**1**] are based on the bound

$$T(x,p) = O\left(\frac{x \log\log x}{p}\right) \tag{1}$$

which is a partial case of [**4**, Theorem 3.5].

Here we obtain an upper bound on $T(x,q)$, albeit weaker than (1), and we follow the approach of [**1**] to estimate the sums $S_a(x,q)$.

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