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THE FUNDAMENTAL THEOREM OF PROJECTIVE GEOMETRY FOR AN ARBITRARY LENGTH TWO MODULE

MARCEL WILD

ABSTRACT. Let V be an arbitrary R-module of length 2 with $n \geq 3$ submodules of length 1. Then every permutation of the length 1 submodules is induced by an isomorphism $V \xrightarrow{\sim} V$ if and only if $n = 3$ or 4.

1. Introduction. In this note all rings R have an identity and all R-modules V are unital. We write $\mathcal{L}(V)$ for the lattice of all submodules of V. Every module isomorphism $f : V \xrightarrow{\sim} V$ clearly induces a lattice isomorphism $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ where $F(W) := f(W)$. Call V *linearly induced* if conversely for each lattice isomorphism $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ there is a module isomorphism $f : V \xrightarrow{\sim} V$ such that $F(W) = f(W)$ for all $W \in \mathcal{L}(V)$. A variant of the *fundamental theorem of projective geometry* can be phrased as follows:

Theorem 1 [**1**, p. 62]. *Let K be a division ring such that every automorphism is inner. Then each K-vector space of finite dimension ≥ 3 is linearly induced.*

(In the classic fundamental theorem of projective geometry [**1**, p. 44] there is *no restriction* on the division ring but then the lattice isomorphism $F : \mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ perhaps is only induced by a *semilinear* bijection $f : V \to V$. We do not wish to bother about semilinearity in this article.)

In particular, in Theorem 1 division rings without proper automorphisms, such as $K = \mathbf{R}$, comply. The lattice $\mathcal{L}(V)$ of subspaces of the K-vector space V is often called the *projective geometry* associated with K. The dimension $1, 2, 3$ subspaces are the *points, lines, planes* of the projective geometry. Lattice isomorphisms $\mathcal{L}(V) \xrightarrow{\sim} \mathcal{L}(V)$ are called

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