ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 6, 2006

LOCALLY EUCLIDEAN METRICS ON S^2 IN WHICH SOME OPEN BALLS ARE NOT CONNECTED

YOUNG DEUK KIM

ABSTRACT. Let $S_r^2 \subset \mathbf{R}^3$ be the 2-sphere with center O and radius r. For all $0 < s \leq 1$, we define a locally Euclidean metric d^s on S_r^2 which is equivalent to the Euclidean metric. These metrics are invariant under Euclidean isometries, and if $0 < s < 1$ then some open balls in (S_r^2, d^s) are not connected.

1. Introduction. Let $S_r^2 \subset \mathbf{R}^3$ be the 2-sphere with center $O = (0,0,0)$ and radius $r > 0$. We write d_E to denote the Euclidean metric on S_r^2. A metric d on the set S_r^2 is called *locally Euclidean* if, for all $P \in S_r^2$, there exists $t > 0$ such that

$$d(Q,R) = d_E(Q,R) \quad \text{for all} \quad Q, R \in B_t(P) = \{S \in S_r^2 \mid d(P,S) < t\}.$$

As usual, two metrics d_1 and d_2 on the set S_r^2 are called *equivalent* if the identity mapping of (S_r^2, d_1) onto (S_r^2, d_2) is a homeomorphism. Notice that the following trivial metric d_T is locally Euclidean but not equivalent to d_E.

$$d_T(P,Q) = \begin{cases} 0 & \text{if } P = Q \\ 1 & \text{if } P \neq Q. \end{cases}$$

In this paper we define a locally Euclidean metric d^s, which is equivalent to d_E and invariant under Euclidean isometries. Notice that the Euclidean metric d_E is trivially locally Euclidean. In fact, the metric d^1 will turn out to be the Euclidean metric d_E. Every open ball in (S_r^2, d_E) is connected. However, if $0 < s < 1$, then some open balls in (S_r^2, d^s) are not connected.

Suppose that $0 < s \leq 1$. Let $-P$ denote the antipodal point of $P \in S_r^2$. Let

$$\alpha = \sin^{-1}\left(\frac{\sqrt{2-s^2}-s}{2}\right), \quad \text{where} \quad 0 \leq \alpha < \pi/4.$$

Received by the editors on September 17, 2003.