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AN EIGENVALUE PROBLEM FOR QUASILINEAR SYSTEMS

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ABSTRACT. The paper deals with the existence of positive solutions for the n-dimensional quasilinear system $(\mathbf{\Phi}(\mathbf{u}'))' + \lambda\mathbf{h}(t)\mathbf{f}(\mathbf{u}) = 0$, $0 < t < 1$, with the boundary condition $\mathbf{u}(0) = \mathbf{u}(1) = 0$. The vector-valued function $\mathbf{\Phi}$ is defined by $\mathbf{\Phi}(\mathbf{u}) = (\varphi(u_1), \dots, \varphi(u_n))$, where $\mathbf{u} = (u_1, \dots, u_n)$, and φ covers the two important cases $\varphi(u) = u$ and $\varphi(u) = |u|^{p-2}u$, $p > 1$, $\mathbf{h}(t) = \operatorname{diag}[h_1(t), \dots, h_n(t)]$ and $\mathbf{f}(\mathbf{u}) = (f^1(\mathbf{u}), \dots, f^n(\mathbf{u}))$. Assume that f^i and h_i are nonnegative continuous. For $\mathbf{u} = (u_1, \dots, u_n)$, let $f_0^i = \lim_{\|\mathbf{u}\|\to 0} f^i(\mathbf{u})/\varphi(\|\mathbf{u}\|)$, $f_\infty^i = \lim_{\|\mathbf{u}\|\to\infty} f^i(\mathbf{u})/\varphi(\|\mathbf{u}\|)$, $i = 1, \dots, n$, $\mathbf{f}_0 = \max\{f_0^1, \dots, f_0^n\}$ and $\mathbf{f}_\infty = \max\{f_\infty^1, \dots, f_\infty^n\}$. We prove that the boundary value problem has a positive solution, for certain finite intervals of λ, if one of $\mathbf{f}_0$ and $\mathbf{f}_\infty$ is large enough and the other one is small enough. Our methods employ fixed point theorems in a cone.

1. Introduction. In this paper we consider the eigenvalue problem for the system

$$(\mathbf{\Phi}(\mathbf{u}'))' + \lambda\mathbf{h}(t)\,\mathbf{f}(\mathbf{u}) = 0, \quad 0 < t < 1, \tag{1.1}$$

with one of the following three sets of the boundary conditions,

$$\mathbf{u}(0) = \mathbf{u}(1) = 0, \tag{1.2a}$$

$$\mathbf{u}'(0) = \mathbf{u}(1) = 0, \tag{1.2b}$$

$$\mathbf{u}(0) = \mathbf{u}'(1) = 0, \tag{1.2c}$$

where $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{\Phi}(\mathbf{u}) = (\varphi(u_1), \dots, \varphi(u_n))$, $\mathbf{h}(t) = \operatorname{diag} \times [h_1(t), \dots, h_n(t)]$ and $\mathbf{f}(\mathbf{u}) = (f^1(u_1, \dots, u_n), \dots, f^n(u_1, \dots, u_n))$. We understand that $\mathbf{u}$, $\mathbf{\Phi}$ and $\mathbf{f}(\mathbf{u})$ are (column) n-dimensional vector-valued functions. Equation (1.1) means that

$$\begin{cases} (\varphi(u_1'))' + \lambda h_1(t)\, f^1(u_1, \dots, u_n) = 0, & 0 < t < 1 \\ \quad\vdots \\ (\varphi(u_n'))' + \lambda h_n(t)\, f^n(u_1, \dots, u_n) = 0, & 0 < t < 1 \end{cases} \tag{1.3}$$

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