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GREEN'S THEOREM WITHOUT DERIVATIVES

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ABSTRACT. The result is established for a Jordan measurable region with rectifiable boundary. The entity F to be integrated by the new plane integral is a function of axis-parallel rectangles, finitely additive on non-overlapping ones, hence unambiguously defined and additive on "figures," i.e., finite unions of axis-parallel rectangles. Define its integral over Jordan measurable S as the limit of its value on the figures, which contain a subfigure of S and are contained in a figure containing S, as the former/complements of the latter expand directedly to fill out S/the complement of S. The integral over every Jordan measurable region exists when additive F is "absolutely continuous" in the sense of converging to zero as the area enclosed by its argument does, or with F the circumferential line integral $\oint P\,dx + Q\,dy$ for P, Q continuous at the rectifiable boundary of S and integrable along axis-parallel line segments. Thus, the equality of this area integral with the line integral around the boundary, to be proved, follows for the various integrals of divergence presented in: *The Riemann approach to integration*, W.F. Pfeffer, Cambridge University Press, New York, 1993.

1. Introduction. In advanced calculus texts, Green's theorem is presented for continuous vector fields with continuous first partial derivatives, defined in a region containing a simple piecewise smooth curve enclosing an area of not too complicated shape. More careful treatments, e.g., [**1**, Sections 10–14], dispense with the continuity of the derivatives in favor of their (bounded existence and) integrability over the interior; recently this requirement has been successively weakened further to integrability of the partials in the "generalized Riemann" sense [**6**, subsection 7.12] and beyond to "gage integrability" [**7**] which even follows from the mere existence of the derivative. By modifying this last integral further, it proves possible to obtain the theorem for a continuous vector field with no differentiability assumption whatsoever.

The "integral" of an additive rectangle function over a Jordan measurable set. For plane Jordan content, see [**1**, subsection

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