ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 1, 2007

THE LENGTH OF DUCCI'S FOUR-NUMBER GAME

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ABSTRACT. The length of a Ducci 4-number game, defined immediately below, is at most six if the initial vector is not cyclically monotone. If it is cyclically monotone, then the length is shown here to be, with error at most 5.4, a linear function of the logarithm of the Euclidean distance from the initial vector (normalized) to the unique normalized vector of infinite length.

1. The main theorem. Define $T : \mathbf{R}^4 \to \mathbf{R}^4$ by the formula

$$T(a, b, c, d) = (|b - a|, |c - b|, |d - c|, |d - a|).$$

For each $v \in \mathbf{R}^4$ the sequence $(T^n(v))_{n \geq 0}$ is called a *Ducci sequence*, or 4-*number game*. A minimal $n \geq 0$ with $T^n(v) = 0$ is called the *length* of v, or of the 4-number game; if there is no such n, then v is said to have *infinite length*. Lengths of 4-number games, and more generally of n-number games, have been one of the major topics in the study of Ducci sequences. (See any of the references below, but especially [**1, 4, 6, 7**].) In this paper we give a simple method for estimating with error less than 5.4 the length of any 4-number game.

As in [**6**], a vector $v \in \mathbf{R}^4$ is said to be *equivalent* to $w \in \mathbf{R}^4$ if w is in the orbit of v under the action on $\mathbf{R}^4$ of the group generated by the permutations R, S, A_f (for $f \in \mathbf{R}$) and M_e (for $0 \neq e \in \mathbf{R}$) of $\mathbf{R}^4$ defined by the formulas $R(a, b, c, d) = (d, c, b, a)$, $S(a, b, c, d) = (d, a, b, c)$, $M_e(v) = ev$, and $A_f(v) = v + (f, f, f, f)$. We also call v *normalized* if it has the form $(0, x, y, 1)$ where $0 < x < y \leq 1 - x$.

In Section 3 we will see that the problem of estimating length reduces to the case of normalized vectors. (Briefly, if a vector is cyclically monotone, then its length equals that of an equivalent normalized vector, and otherwise the length is at most 6 and hence is known to within 5.4.) We now state our main theorem and give a simple estimate

2000 AMS *Mathematics Subject Classification.* Primary 11B83, Secondary 11B37.
Key words and phrases. Ducci sequence, four-number game, length.
Received by the editors on April 12, 2004, and in revised form on July 2, 2004.