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A GENERALIZATION OF KUMMER'S IDENTITY

RAIMUNDAS VIDUNAS

ABSTRACT. The well-known formula of Kummer evaluates the hypergeometric series ${}_2F_1\left(\begin{smallmatrix}A,B\\C\end{smallmatrix}\middle|-1\right)$ when the relation $C-A+B=1$ holds. This paper deals with the evaluation of ${}_2F_1(-1)$ series in the case when $C-A+B$ is an integer. Such a series is expressed as a sum of two Γ-terms multiplied by terminating ${}_3F_2(1)$ series. A few such formulas were essentially known to Whipple in the 1920s. Here we give a simpler and more complete overview of this type of evaluation. Additionally, algorithmic aspects of evaluating hypergeometric series are considered. We illustrate Zeilberger's method and discuss its applicability to nonterminating series and present a couple of similar generalizations of other known formulas.

1. The generalization. The subject of this paper is a generalization of Kummer's identity (see [**11**], [**2**, Section 2.3] or [**1**, Corollary 3.1.2]):

$$(1)\qquad {}_2F_1\left(\begin{matrix}a,b\\1+a-b\end{matrix}\middle|-1\right)=\frac{\Gamma(1+a-b)\Gamma(1+\frac{a}{2})}{\Gamma(1+a)\Gamma(1+\frac{a}{2}-b)}.$$

The hypergeometric series on the left is defined if $a-b$ is not a negative integer, and it is absolutely convergent for $\operatorname{Re}(b)<1/2$. After analytic continuation of ${}_2F_1\left(\begin{smallmatrix}a,b\\1+a-b\end{smallmatrix}\middle|z\right)$ on $\mathbf{C}\setminus[1,\infty)$, and after division of both sides by $\Gamma(1+a-b)$ the formula has meaning and is correct for all complex a,b. In this paper, whenever ${}_2F_1\left(\begin{smallmatrix}A,B\\C\end{smallmatrix}\middle|z\right)$ denotes a well-defined hypergeometric series, it also denotes its analytic continuation on $\mathbf{C}\setminus[1,\infty)$.

The generalization to be considered evaluates the hypergeometric series ${}_2F_1\left(\begin{smallmatrix}A,B\\C\end{smallmatrix}\middle|-1\right)$ whenever $C-A+B$ is any integer. In the terminology of [**1**], our generalization applies to ${}_2F_1(-1)$ series that are *contiguous* to a series for Kummer's formula (1). As is known (see [**1**,

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