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# LEONARD PAIRS FROM 24 POINTS OF VIEW

PAUL TERWILLIGER

ABSTRACT. Let $\mathbf{K}$ denote a field and let $V$ denote a vector space over $\mathbf{K}$ with finite positive dimension. We consider a pair of linear transformations $A : V \to V$ and $A^* : V \to V$ that satisfy both conditions below:

(i) There exists a basis for $V$ with respect to which the matrix representing $A$ is diagonal and the matrix representing $A^*$ is irreducible tridiagonal.

(ii) There exists a basis for $V$ with respect to which the matrix representing $A^*$ is diagonal and the matrix representing $A$ is irreducible tridiagonal.

We call such a pair a *Leonard pair* on $V$. Referring to the above Leonard pair, we investigate 24 bases for $V$ on which the action of $A$ and $A^*$ take an attractive form. Our bases are described as follows. Let $\Omega$ denote the set consisting of four symbols $0, d, 0^*, d^*$. We identify the symmetric group $S_4$ with the set of all linear orderings of $\Omega$. For each element $g$ of $S_4$, we define an (ordered) basis for $V$, which we denote by $[g]$. The 24 resulting bases are related as follows. For all elements $wxyz$ in $S_4$, the transition matrix from the basis $[wxyz]$ to the basis $[xwyz]$, (respectively $[wyxz]$), is diagonal, (respectively lower triangular). The basis $[wxzy]$ is the basis $[wxyz]$ in inverted order. The transformations $A$ and $A^*$ act on the 24 bases as follows: For all $g \in S_4$, let $A^g$, (respectively $A^{*g}$), denote the matrix representing $A$, (respectively $A^*$), with respect to $[g]$. To describe $A^g$ and $A^{*g}$, we refer to $0^*, d^*$ as the *starred* elements of $\Omega$. Writing $g = wxyz$, if neither of $y, z$ are starred then $A^g$ is diagonal and $A^{*g}$ is irreducible tridiagonal. If $y$ is starred but $z$ is not, then $A^g$ is lower bidiagonal and $A^{*g}$ is upper bidiagonal. If $z$ is starred but $y$ not, then $A^g$ is upper bidiagonal and $A^{*g}$ is lower bidiagonal. If both of $y, z$ are starred, then $A^g$ is irreducible tridiagonal and $A^{*g}$ is diagonal.

We define a symmetric binary relation on $S_4$ called adjacency. An element $wxyz$ of $S_4$ is by definition adjacent to each of $xwyz$, $wyxz$, $wxzy$ and no other elements of $S_4$. For all ordered pairs of adjacent elements $g, h$ in $S_4$, we find the entries of the transition matrix from the basis $[g]$ to the basis

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