

## MULTIPLE GAMMA FUNCTION, ITS $q$ - AND ELLIPTIC ANALOGUE

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**ABSTRACT.** Vignéras's multiple gamma function is introduced as a function satisfying a generalization of the Bohr-Mollerup theorem. An infinite product representation and an asymptotic expansion of the function are given. Furthermore, its  $q$ - and elliptic analogue are introduced as relevant with the defining relations of  $q$ -gamma function and of elliptic gamma function.

**1. Introduction.** In 1904, Barnes [4, 5, 6, 7] introduced his multiple gamma function by using the multiple Hurwitz zeta functions. After his discovery, Hardy [14, 15] studied this function from his viewpoint of the theory of elliptic functions, and Shintani [37, 38] applied it to the study on the Kronecker limit formula for zeta functions attached to certain algebraic fields. At the end of the 70's, Vignéras [45] redefined the multiple gamma function as the function satisfying a generalization of the Bohr-Mollerup theorem. Vignéras [45], Voros [46], Vardi [44] and Kurokawa [22–25] showed that it is applicable to represent gamma factors of the Selberg zeta functions of compact Riemann surfaces and the determinants of the Laplacians on some Riemannian manifolds. Recently, multiple gamma functions have been applied to studies on the Kniznik-Zamolodchikov equation with  $|q| = 1$  [18, 26, 27], on eigenfunctions of commuting difference operators [36] and on  $q$ -analysis with  $|q| = 1$  [29, 33, 40].

In the theory of special functions, constructions of  $q$ -analogue and of elliptic analogue are known as generalizations in the other direction.  $q$ -Analogue of Euler's gamma function was introduced by Jackson [16, 17]. This function plays essential roles in  $q$ -analysis. One of the most significant features of the function was pointed out by Askey [3]. He remarked that the  $q$ -gamma function satisfies a  $q$ -analogue of the Bohr-Mollerup theorem. Considering Askey's result together with Vignéras' one, it would be natural to consider the construction of functions satisfying a  $q$ -analogue of the generalized Bohr-Mollerup theorem. On

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