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GENERALIZED UMEMURA POLYNOMIALS

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ABSTRACT. We introduce and study generalized Umemura polynomials $U_{n,m}^{(k)}(z,w;a,b)$ which are a natural generalization of the Umemura polynomials $U_n(z,w;a,b)$ related to Painlevé VI equation. We will show that if $a=b$ or $a=0$ or $b=0$, then polynomials $U_{n,m}^{(0)}(z,w;a,b)$ generate solutions to Painlevé VI. We will describe a connection between polynomials $U_{n,m}^{(0)}(z,w;a,0)$ and certain Umemura polynomials $U_k(z,w;\alpha,\beta)$.

1. Introduction. There is a vast body of literature devoted to the Painlevé VI equation $P_{\mathrm{VI}} := P_{\mathrm{VI}}(\alpha,\beta,\gamma,\delta)$:
(1.1)
$$\frac{d^2q}{dt^2} = \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right) + \frac{q(q-1)(q-t)}{t^2(t-1)^2}\left(\alpha-\beta\frac{t}{q^2}+\gamma\frac{(t-1)}{(q-1)^2}+\delta\frac{t(t-1)}{(q-t)^2}\right)$$

where $t\in\mathbf{C}$, $q := q(t;\alpha,\beta,\gamma,\delta)$ is a function of t and $\alpha,\beta,\gamma,\delta$ are arbitrary complex parameters. It is well known and goes back to Painlevé that any solution $q(t)$ of the equation P_{VI} satisfies the so-called Painlevé property:

• the critical points 0, 1 and ∞ of the equation (1.1) are the only *fixed singularities* of $q(t)$.

• any *movable singularity* of $q(t)$, the position of which depends on integration constants, is a pole.

In this paper we introduce and initiate the study of certain special polynomials related to the Painlevé VI equation, namely, the generalized Umemura polynomials $U_{n,m}^{(k)}(z,w;a,b)$. These polynomials have many interesting combinatorial and algebraic properties and in the particular case $n=0=k$ coincide with Umemura's polynomials $U_m(z^2,w^2;a,b)$,

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