

PARTIAL DIFFERENTIAL EQUATIONS SATISFIED BY POLYNOMIALS WHICH HAVE A PRODUCT FORMULA

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ABSTRACT. The classical families of orthogonal polynomials arise as eigenfunctions of Sturm-Liouville problems. In 1929, Bochner addressed the converse question: Which linear second order differential operators can have an infinite family of polynomials, \mathcal{P} , as their eigenfunctions? The classification that he gave showed that there were, up to a linear change of variables, a unique differential operator associated with each such family, and the argument did not even require that the members of \mathcal{P} were orthogonal, only that there were “enough” polynomials in the family (indeed, in some cases, the polynomials are not orthogonal). In this paper it will be shown that certain families of bivariate polynomials satisfy not just one but a pair $L^{(1)}$ and $L^{(2)}$ of partial differential operators with bounds on the order of the operators determined by properties of \mathcal{P} . We also give a simple condition that the polynomials be completely determined by the pair of operators (up to multiplicative constants). This article includes detailed discussions of five examples of such polynomial families. We shall also discuss $\Delta(\mathcal{P})$, the algebra of operators which have \mathcal{P} as eigenfunctions, and we give sufficient conditions that every member of $\Delta(\mathcal{P})$ is given uniquely as a polynomial in $L^{(1)}$ and $L^{(2)}$. This will be the case in all five examples.

1. Introduction. The classical families of orthogonal polynomials arise as eigenfunctions of Sturm-Liouville problems. In 1929, Bochner addressed the converse question [2]: Which linear second order differential operators can have an infinite family of polynomials, \mathcal{P} as their eigenfunctions? The classification that he gave showed that there were, up to a linear change of variables, a unique differential operator associated with each such family, and the argument did not even require that the members of \mathcal{P} were orthogonal, only that there were “enough”

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