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DIRICHLET SPLINES AS FRACTIONAL INTEGRALS OF B-SPLINES

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ABSTRACT. Using *Dirichlet averages* we generalize the notion of a classical divided difference of a function by introducing a parameter $\mathbf{r}$ in $\mathbf{R}_+^{k+1}$. The case $\mathbf{r}$ in $\mathbf{N}^{k+1}$ is related to divided differences with multiple knots. We give an interpretation of these generalized differences in terms of fractional operators applied to classical divided differences considered as functions of their knots. The result is then applied to show that *Dirichlet splines* can be seen as fractional derivatives of B-splines.

1. Introduction. Splines are a well-established class of functions in many fields of applied analysis. Their properties, for example, allow good approximations and efficient algorithms for computation. In statistics, splines have an even older history. They occur as density functions of multivariate probability measures. Both fields seemed to have been unaware of their common interest in spline functions up into the 1980s. In 1986 papers by Dahmen and Micchelli [**7**] and Karlin, Micchelli and Rinott [**10**] appeared which began to point out these connections.

One of the key concepts in approximation with spline functions is a convenient basis for the underlying function space, the so-called *B-splines*. B-splines are nonnegative, compactly supported and are solutions to certain minimalization problems, just to mention a few of their central properties. From the statistical point of view, spline functions occur in connection with the uniform distribution over the standard simplex. To investigate more general classes of distributions over the simplex, density functions have been introduced in the defining equation for spline densities, e.g., the Dirichlet distribution and the Gamma distribution, respectively. This has led to so-called *Dirichlet*

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