

## ORTHOGONAL POLYNOMIALS WITH RESPECT TO A DIFFERENTIAL OPERATOR. EXISTENCE AND UNIQUENESS

A.I. APTEKAREV, G.T. LÓPEZ LAGOMASINO AND F. MARCELLÁN

ABSTRACT. A new type of orthogonal polynomial connected with linear differential operators, intimately related with Sobolev orthogonal polynomials and Hermite-Padé polynomials, is introduced. We study the question of uniqueness of the sequence of orthogonal polynomials arising from this construction. As we show, this problem is related to the analytic properties of the fundamental system of solutions of the operator. The notion of  $T$ -system of Tchebyshev plays a key role in the analysis. Some examples of general classes of operators which produce a unique system of polynomials are given.

### 1. Introduction.

1.1 *Definition of the main object.* In the last two decades there has been a growing interest in different generalizations of the notion of orthogonal polynomials. To name a few, we have *Hermite-Padé polynomials*, [2, 10], and the *matrix orthogonal polynomials*, [1, 3]. They arise in a natural way in problems of approximation theory, mathematical physics and number theory. From the theoretical point of view, the study of these constructions pose questions whose solution requires new methods and interrelations of classical techniques of analysis.

In this paper we introduce sequences of polynomials orthogonal with respect to a linear homogeneous differential operator.

*Definition.* Let  $\sigma(x)$  be a positive Borel measure on the real line and  $\{\rho_k(x)\}_{k=0}^m$ ,  $\rho_m \equiv 1$ , be a set of functions such that  $\rho_k(x) d\sigma(x)$  has

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