ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 2002

FLUCTUATION OF SECTIONAL CURVATURE FOR CLOSED HYPERSURFACES

MARIUS OVERHOLT

ABSTRACT. Liebmann proved in 1899 that the only closed surfaces in Euclidean three-space that have constant Gauss curvature are round spheres. Thus, if a closed surface in three-space is not a topological sphere, its Gauss curvature must fluctuate. We consider quantitative formulations of this fact, also in higher dimensions.

0. Introduction. Consider a smooth closed manifold M of dimension n which has an immersion $f : M \to (\mathbf{R}^{n+1}, \mathrm{can})$ as a hypersurface in Euclidean space. The immersion pulls back the canonical Riemannian metric on $\mathbf{R}^{n+1}$ to a Riemannian metric on M, called the induced metric, which we denote by $f^*\mathrm{can}$. If M is not diffeomorphic to S^n, the sectional curvature of $f^*\mathrm{can}$ must fluctuate. For if the sectional curvature is constant, it must be positive. Then the shape operator is everywhere definite, so the hypersurface is diffeomorphic to S^n by a theorem of Hadamard.

We seek a lower bound for the amount of fluctuation of sectional curvature, dependent on M, but independent of the particular immersion f as far as possible. For any closed Riemannian manifold, the set of values of the sectional curvature forms a closed bounded interval. The task at hand is to give a lower bound for the length $l(\mathrm{sec})$ of this interval for the Riemannian metrics $f^*\mathrm{can}$. Because of scaling, it is clear that such a bound cannot depend on M alone, but must have some dependence on the immersion f. It turns out that it is possible to give a lower bound depending only on the topology of M and its volume with respect to $f^*\mathrm{can}$.

1. Fluctuation of sectional curvature. Let F be some fixed field, and $\beta_j(M;F) = \dim H_j(M;F)$ the Betti numbers of M with respect to the field F and $\beta(M;F)$ their sum. Then $l(\mathrm{sec})$ can be estimated from below by $\mathrm{vol}\,(M)$ and $\beta(M;F)$.

Received by the editors on October 2, 2000.