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NORMAL HYPERBOLICITY FOR FLOWS AND NUMERICAL METHODS

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ABSTRACT. In this paper we prove that normally hyperbolic invariant manifolds persist between flows and numerical methods in both directions. This means that normal hyperbolicity of flows is preserved under numerical methods and that normally hyperbolicity for numerical methods is inherited by flows.

1. Definitions and statement of theorems. Let M be a smooth complete Riemannian manifold with a distance d arising from the Riemannian metric and $\mathrm{Diff}\,(M)$ be the set of diffeomorphisms on M with the strong topology and distance d_{C^1}. A *flow* is a map $\varphi : \mathbf{R}\times M \to M$ that satisfies the group property: $\varphi^s(\varphi^t(x)) = \varphi^{s+t}(x)$.

Definition 1. For $p \geq 1$, let φ be a C^{p+1} flow on M. A C^{p+1} function $N : \mathbf{R} \times M \to M$ is called a *numerical method of order* p for φ^t if there are positive constants K and h_0 such that $d(\varphi^h(x), N^h(x)) \leq Kh^{p+1}$, for all $h \in [0, h_0]$ and $x \in M$. Here h stands for a stepsize of N. We denote the i-th iterate of $N^h(x)$ by $(N^h)^i(x)$.

Numerical methods arise from computer simulation and numerical approximation. For instance, both explicit and implicit Runge-Kutta methods satisfy the above conditions (see [**1**]).

It is well known that the time-h map of the flow and the numerical method of stepsize h are C^1 close polynomially in terms of h.

Lemma 1 [**6**]. *Let N be a numerical method of order p for a C^{p+1} flow φ on a compact manifold M. Then there is a constant K_1 such that $d_{C^1}(\varphi^h, N^h) \leq K_1 h^p$ for all sufficiently small h. Moreover, given $T > 0$, there is a constant K_2 such that $d_{C^1}(\varphi^T, (N^{T/n})^n) \leq K_2 n^{1-p}$ for all large positive integers n.*

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