ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 32, Number 1, Spring 2002

GROWTH AND COEFFICIENT ESTIMATES FOR UNIFORMLY LOCALLY UNIVALENT FUNCTIONS ON THE UNIT DISK

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ABSTRACT. In this note we shall give a sharp distortion estimate for a uniformly locally univalent holomorphic function on the unit disk in terms of the norm of pre-Schwarzian derivative. As applications, we shall investigate the growth of coefficients and integral means of such a function and mention a connection with Hardy spaces. We also give norm estimates for typical classes of univalent functions.

1. Introduction. We will say that a holomorphic function f on the unit disk $\mathbf{D}$ is *uniformly locally univalent* if f is univalent on each hyperbolic disk $D(a,\rho) = \{z \in \mathbf{D}; |(z-a)/(1-\bar{a}z)| < \tanh\rho\}$ with radius ρ and center $a \in \mathbf{D}$ for a positive constant ρ. In particular, a holomorphic universal covering map of a plane domain D is uniformly locally univalent if and only if the boundary of D is uniformly perfect, see [**18**] or [**22**]. Also it is well known, cf. [**24**], that a holomorphic function f on the unit disk is uniformly locally univalent if and only if the pre-Schwarzian derivative (or nonlinearity) $T_f = f''/f'$ of f is hyperbolically bounded, i.e., the norm

$$\|T_f\| = \sup_{z\in\mathbf{D}}(1-|z|^2)|T_f(z)|$$

is finite. This quantity can be regarded as the Bloch semi-norm of the function $\log f'$. We remark that a holomorphic function f is locally univalent at the point z if and only if $T_f = f''/f'$ is a well-defined holomorphic function near z. Roughly speaking, the quantity T_f measures the deviation of f from orientation-preserving similarities (nonconstant linear functions). In the following it is sometimes essential to consider the semi-norm

$$\|T_f\|_0 = \overline{\lim_{|z|\to 1-0}}(1-|z|^2)|T_f(z)| = 2\overline{\lim_{|z|\to 1-0}}(1-|z|)|T_f(z)|$$

1991 *Mathematics Subject Classification.* Primary 30C45, 30C50, Secondary 30C80.
Key words and phrases. pre-Schwarzian derivative, uniformly locally univalent, growth estimate, coefficient estimate.
Received by the editors on September 15, 2000.