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WHEN IS A POLYGONAL PYRAMID NUMBER AGAIN POLYGONAL?

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ABSTRACT. We consider a Diophantine equation arising from a generalization of the classical Lucas problem of the square pyramid: when is the sum of the first m-gonal numbers n-gonal? We use the theory of elliptic surfaces to deduce several families of parametric solutions of the problem.

1. Introduction. In 1875 Lucas proposed the problem of proving that $1^2 + 2^2 + \cdots + 24^2 = 70^2$ is the only nontrivial solution to the problem referred to as "the cannonball problem" or "the problem of the square pyramid": When is the sum of the first n squares a perfect square? This problem was settled finally by G.N. Watson in 1918 (see [**1**] for the history and an elementary proof of the problem).

In the present paper, by regarding a perfect square as 4-gonal number, we consider the following generalization of the cannonball problem: When is the sum of the first m-gonal numbers once again an m-gonal number, or more generally, a polygonal number of possibly different order n? Here, for positive integers $m \geq 3$ and $i \geq 1$, the ith *polygonal number of order* m or the ith m*-gonal number*, is given by

$$G_m(i) = \frac{m-2}{2}i^2 - \frac{m-4}{2}i. \tag{1}$$

We call the sum of the first i m-gonal numbers the ith polygonal pyramid number of order m, or the ith m*-gonal pyramid number* and denote it by $\mathrm{Pyr}_m(i) : \mathrm{Pyr}_m(i) = \sum_{j=1}^{i} G_m(j)$. Then our problem of the polygonal pyramid is to find (positive) integer solutions (x, y) to the equation

$$G_n(y) = \mathrm{Pyr}_m(x) \tag{2}$$

for fixed integers $m, n \geq 3$. By (1), we can write this equation explicitly as

$$3(n-2)y^2 - 3(n-4)y = (m-2)x^3 + 3x^2 - (m-5)x. \tag{3}$$

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