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# ON THE SINGULARITIES AT INFINITY OF PLANE ALGEBRAIC CURVES

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ABSTRACT. We study polynomials in two complex variables with no critical points and with at most one irregular value at infinity. We give some applications to polynomial automorphisms.

**Introduction.** Let $f : \mathbf{C}^2 \to \mathbf{C}$ be a polynomial of degree $d > 1$ with finite set of critical points, i.e., such that the partial derivatives $(\partial f/\partial X)$, $(\partial f/\partial Y)$ do not have common factors. Then the polynomials $f - t$, $t \in \mathbf{C}$ have no multiple factors.

Let $C^t$ be the projective closure of the fiber $f^{-1}(t)$. If $F(X,Y,Z)$ is the homogeneous form corresponding to $f = f(X,Y)$, then $C^t$ is given by the equation $F(X,Y,Z) - tZ^d = 0$. Let $L_\infty \subset \mathbf{P}^2(\mathbf{C})$ be the line at infinity defined by $Z = 0$, and let $C_\infty = C^t \cap L_\infty$. Then the set $C_\infty$ is described by equations $F(X,Y,0) = Z = 0$ in $\mathbf{P}^2(\mathbf{C})$. For every point $p \in C_\infty$, we consider the Milnor number $\mu_p^t = \mu_p(C^t)$, and we put $\mu_p^{\min} = \inf_{t\in\mathbf{C}} \mu_p^t$. The set $\Lambda(f) = \{t \in \mathbf{C} : \mu_p^t > \mu_p^{\min} \text{ for a } p \in C_\infty\}$ is finite (see [**6**]). The elements of $\Lambda(f)$ are called irregular values of $f$. We put according to Broughton $\lambda^t(f) = \sum_{p\in C_\infty} (\mu_p^t - \mu_p^{\min})$ and $\lambda(f) = \sum_{t\in\mathbf{C}} \lambda^t(f)$.

Equivalent definitions of irregular values are discussed in [**10**]. A polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ is called a coordinate polynomial if there is a polynomial $g : \mathbf{C}^2 \to \mathbf{C}$ such that $\mathbf{C}[X,Y] = \mathbf{C}[f,g]$. The famous Abhyankar-Moh theorem [**2**] can be stated as follows: *an affine plane curve is isomorphic to a line if and only if its minimal equation is a coordinate polynomial.* Using the Abhyankar-Moh theorem, Ephraim proved [**11**] that a polynomial $f : \mathbf{C}^2 \to \mathbf{C}$ is a coordinate polynomial if and only if $f$ has no critical points and $\Lambda(f) = \varnothing$.

In this note we study polynomials in two complex variables with no critical points in $\mathbf{C}^2$. Our aim is to characterize polynomials with one

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