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ABELIAN GROUPS WITH SELF-INJECTIVE QUASI-ENDOMORPHISM RINGS

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1. Introduction. One of the most important concepts in the discussion of endomorphism rings of torsion-free abelian groups is that of faithfulness. A left R-module A is *fully faithful* (*faithful*) if $M \otimes_R A \neq 0$ for all (finitely generated) right R-modules M. It is easy to see that a faithful module which is flat is fully faithful. On the other hand, faithful modules exist which are not fully faithful: $\oplus_p \mathbf{Z}/p\mathbf{Z}$ where the direct sum is taken over all primes is a faithful $\mathbf{Z}$-module, but is not fully faithful.

Abelian groups which are faithful or fully faithful as modules over their endomorphism ring share some of the homological properties of torsion-free groups of rank 1 which Baer discussed in 1937 in [**11**]. For instance, a self-small abelian group A is (faithful) fully faithful as a module over its endomorphism ring if and only if an exact sequence $0 \to B \xrightarrow{\alpha} G \to P \to 0$ splits if P is A-projective (of finite A-rank) and $G = S_A(G) + \alpha(B)$, for details, see [**2**] and [**9**]. Here P is *A-projective* (*of finite A-rank*) if it is a direct summand of $\oplus_J A$ for some (finite) index-set J, and $S_A(G) = \mathrm{Hom}\,(A, G)A$. The group A is *self-small* if, for every index-set I and all $\alpha \in \mathrm{Hom}\,(A, \oplus_I A)$ there is a finite subset I' of I such that $\alpha(A) \subseteq \oplus_{I'} A$. For example, every torsion-free group of finite rank is self-small, but self-small torsion-free groups of arbitrary cardinality exist.

In this paper the concept of faithfulness is extended to the quasi-category of torsion-free abelian groups: A torsion-free group A is said to be *quasi-fully faithful* if $\mathbf{Q}A = \mathbf{Q} \otimes_{\mathbf{Z}} A$ is a fully faithful $\mathbf{Q}E$-module where $E = E(A)$ denotes the *endomorphism ring of* A and $\mathbf{Q}E = \mathbf{Q} \otimes_{\mathbf{Z}} E$ is its *quasi-endomorphism ring*. Theorem 2.1 and its corollaries give additional characterizations of quasi-fully faithful groups. It is shown that every quasi-fully faithful group A has the finite *quasi-Baer-splitting property*, i.e., an exact sequence $0 \to B \xrightarrow{\alpha}$

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