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A SYSTEMATIC GENERALIZATION PROCEDURE FOR FIXED-POINT THEOREMS

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ABSTRACT. The progress of mathematics often follows a standard path: the discovery of a new theorem, followed by a systematic exploration of that theorem. Two standard ways of exploring theorems are by weakening the hypotheses and strengthening the conclusion. This paper discusses another way to explore theorems through the sharing of hypotheses, and develops this idea in the context of three classical fixed-point theorems: the Banach Contraction Principle, the Tarski Fixed-Point Theorem for complete lattices and the Brouwer Fixed-Point Theorem for solid n-spheres.

Introduction. Mathematical research often progresses in accordance with the anthropologist Steven Jay Gould's theory of 'punctuated equilibrium,' in which dramatic breakthroughs are interspersed with long periods of quiet but gradual advance.

The Banach Contraction Principle provides a good example. When first discovered, this theorem represented a dramatic breakthrough. We state a preliminary version here as a reference point.

Banach Contraction Principle. *Let T be a map of a complete metric space X into itself. Assume there exists a constant $M \in (0,1)$ such that $d(Tx,Ty) \leq Md(x,y)$. Then T has a fixed point.*

Once the dramatic breakthrough has been made, other mathematicians will use the basic idea to discover additional theorems. Two classic ways to go about this are: (1) to strengthen the conclusion and (2) to weaken the hypothesis. In the case of the Banach Contraction Principle, a well-known strengthening of the conclusion is that the fixed point is unique. There are many examples of weakening the hypothesis; one such is Browder's result [**2**] that the conclusion holds if one assumes that u is an upper semi-continuous function on the positive reals such that $u(t) < t$ and $d(Tx,Ty) < u(d(x,y))$.

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