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# HARMONIC MAPPINGS RELATED TO SCHERK'S SADDLE-TOWER MINIMAL SURFACES

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**1. Introduction.** A *harmonic mapping* is a complex-valued univalent harmonic function defined in some domain of the complex plane. Harmonic mappings are of interest in differential geometry because they provide isothermal coordinates for nonparametric minimal surfaces, leading to the classical Weierstrass-Enneper representation in terms of analytic functions. (See, for instance, [**8**], [**9**], [**2**], [**6**], [**4**].) More recently, harmonic mappings have been studied from the viewpoint of complex analysis, as generalizations of conformal mappings, see [**1**], [**3**].

The purpose of this note is to investigate a family of harmonic mappings that arise in connection with Scherk's classical "saddle-tower" minimal surface (see [**2**] or [**9**]) and its generalizations recently found by Karcher [**7**]. The mappings in question are defined on the unit disk **D** by

$$F_n(z) = -\frac{2}{n}\sum_{k=1}^{n} \alpha^k \log|z-\alpha^k|, \quad n = 3, 4, \dots, \tag{1}$$

where $\alpha = e^{2\pi i/n}$ is a primitive $n$th root of unity. Each function $F_n$ is clearly harmonic in **D**, but its univalence is not so obvious *a priori*. A direct proof of the univalence will be given in Section 2. Meanwhile, some *Mathematica*-produced images of the disk under $F_n$ are displayed in Figure 1 for $n = 3$, 4, 6, and 10. The figure shows the images of equally spaced concentric circles and radial segments, giving in particular a graphical demonstration of the univalence. The infinite spires correspond to the $n$th roots of unity: $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$. It is clear from the formula (1) that $F_n$ maps each radial segment from 0 to $\alpha^k$ onto the radial half-line in the same direction. In fact, essentially the same geometric argument (pairing symmetric terms of the sum) shows that $F_n$ maps each intermediate segment from 0 to $\alpha^{k-1/2}$ onto a radial segment in the same direction.

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