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THE SPECTRAL GEOMETRY OF RIEMANNIAN SUBMERSIONS FOR MANIFOLDS WITH BOUNDARY

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ABSTRACT. We study the spectral geometry of a Riemannian submersion $\pi : Z \to Y$ where Z and Y are compact Riemannian manifolds with smooth boundaries and where $\pi : \partial Z \to \partial Y$ is also a Riemannian submersion. We impose suitable boundary conditions and give necessary and sufficient conditions that π^* preserve all the eigenforms of the Laplacian. We also study when a single eigenvalue can change.

0. Introduction. All manifolds in this note are assumed to be compact, connected, orientable, smooth Riemannian manifolds with smooth boundaries. Let $\Delta_M^p := \delta_M d_M + d_M \delta_M$ be the Laplace Beltrami operator on the space of smooth p forms $C^\infty \Lambda^p M$ on such a manifold M. We must impose suitable boundary conditions $\mathcal{B}$ if ∂M is nonempty. Section 1 is devoted to a brief review of Dirichlet, Neumann, absolute and relative boundary conditions; these are the boundary conditions that we will consider. Let $\Delta_{M,\mathcal{B}}^p$ be the Laplacian on M with domain defined by the boundary condition $\mathcal{B}$. Denote the corresponding eigenspaces by

$$E(\lambda, \Delta_{M,\mathcal{B}}^p) := \{\Phi \in C^\infty(\Lambda^p M) : \Delta_M^p \Phi = \lambda\Phi \text{ and } \mathcal{B}\Phi = 0\}.$$

In Lemma 1.2 we show $\Delta_{M,\mathcal{B}}^p$ is self-adjoint. If $\mathcal{B}$ denotes Dirichlet, relative, or absolute boundary conditions, $\Delta_{M,\mathcal{B}}^p$ is a nonnegative operator. By contrast, if $\mathcal{B}$ denotes Neumann boundary conditions, then $\Delta_{M,\mathcal{B}}^p$ can have negative spectrum as we shall show in Theorem 4.4. The material of Section 1 is fairly well known; we have organized it for the convenience of the reader in later sections.

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