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EXACT LOCATION OF α-BLOCH SPACES IN L_a^p AND H^p OF A COMPLEX UNIT BALL

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ABSTRACT. In this paper we prove that, on the unit ball of $\mathbf{C}^n$, (i) for $f \in H(B)$ and $0 < \alpha < \infty$, $f \in \mathcal{B}^\alpha \Leftrightarrow \sup_{z\in B} |\mathcal{R}f(z)|(1-|z|^2)^\alpha < \infty$; as a corollary, $\mathcal{B}^\alpha = A(B) \cap \operatorname{Lip}(1-\alpha)$ for $0 < \alpha < 1$. (ii) $B^{\alpha(<1+(1/p))} \subset L_a^p \subset \mathcal{B}^{1+((n+1)/p)}$, $\mathcal{B}^{\alpha(<1)} \subset H^p \subset \mathcal{B}^{1+(n/p)}$ for $n > 1$ and $0 < p < \infty$, where L_a^p, H^p denote the Bergman spaces and Hardy spaces, respectively. And $\mathcal{B}^1 \subset \cap_{0<p<\infty} L_a^p \subset \mathcal{B}^{\alpha(>1)}$, $\mathcal{B}^{\alpha(<1)} \subset \cap_{0<p<\infty} H^p \subset \mathcal{B}^{\alpha(>1)}$. Further, it is proved with constructive methods that all of the above containments are strict and best possible.

1. Introduction. Let $H(B)$ denote the class of all holomorphic functions in the unit ball B of $\mathbf{C}^n$. We say that $f \in \mathcal{B}^\alpha$, α-Bloch, if

$$\|f\|_{\mathcal{B}^\alpha(B)} = \sup_{z\in B} |\nabla f(z)|(1-|z|^2)^\alpha < \infty, \quad 0 < \alpha < \infty.$$

It is clear that $\mathcal{B}^\alpha$ is a normed linear space, modulo constant functions, and $\mathcal{B}^{\alpha_1} \subset \mathcal{B}^{\alpha_2}$ for $\alpha_1 < \alpha_2$. When $n = 1$, replace them by $H(D)$ and $\mathcal{B}^\alpha(D)$, where D denotes the unit disk of complex plane.

Hardy and Littlewood proved that [**3**], [**2**]: $\mathcal{B}^\alpha(D) = \operatorname{Lip}(1-\alpha)$. We know that $\operatorname{Lip}\beta$ can be used to describe the dual space of Hardy space $H^p(D)$ for $0 < p < 1$ [**2**]. So $\mathcal{B}^\alpha$ are important in the theory of Hardy spaces. In [**15**] we gave some invariant gradient characterizations and Bergman-Carleson measure characterization of $\mathcal{B}^\alpha$ on the unit ball.

For $\mathcal{B}^1 = \operatorname{Bloch}(B)$, Timoney showed that $H_p \not\subset \operatorname{Bloch}(B)$ for any $p \in (0,\infty)$, but he did not know whether there were Bloch functions which were not in H^p or not, see Example 3.7(3) of [**12**]. Later on, in [**10**], Ryll and Wojtaszczyk pointed out that $\operatorname{Bloch}(B) \not\subset H^p$;

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