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A THEOREM ON TRANSCENDENCE OF INFINITE SERIES

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1. Introduction. There are a number of sufficient conditions known within the literature for an infinite series, $\sum_{n=1}^{\infty} 1/a_n$, of positive rationals to converge to an irrational number (see [**3**], [**4**], [**11**], [**10**] and the references cited therein). These conditions, which are quite varied in form, share one common feature, namely, they all require rapid growth of the sequence $\{a_n\}$ to deduce irrationality of the series. As an illustration consider the following results of Sándor which have been taken from [**11**] and [**12**].

Theorem 1.1. *Let $\{a_m\}$, $m \geq 1$, be a sequence of positive integers such that*

$$\limsup_{m\to\infty} \frac{a_{m+1}}{a_1 a_2 \cdots a_m} = \infty \quad \textit{and} \quad \liminf_{m\to\infty} \frac{a_{m+1}}{a_m} > 1.$$

Then the sum of the series $\sum_{m=1}^{\infty} 1/a_m$ is an irrational number. Alternatively, if $\{a_m\}$ and $\{b_m\}$ are a sequence of positive integers with $b_m | b_{m+1}$, $b_m \to \infty$ and $\lambda > 2$ exists such that $b_N^{\lambda} \sum_{m>N} a_m/b_m < 1$, for infinitely many N, then the sum of the series $\sum_{m=1}^{\infty} a_m/b_m$, when convergent, is a transcendental number.

In view of the fact that all algebraic numbers cannot be approximated by infinitely many rationals m/n to within $1/n^r$ for any $r \in N\backslash\{0\}$, one possible approach to demonstrating the transcendence of a given series having sum s would be to produce a sequence of rapidly converging rational approximations to s, for example, using the partial sums of the series. Such an approximation, in the absence of methods for accelerating the convergence of a series, may still be achieved if the sequence $\{a_n\}$ has sufficiently strong growth as in Theorem 1.1. In this paper we do precisely this by showing that, under the following growth

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