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APPROACH GROUPS

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ABSTRACT. Any normed vector space X is a topological group with respect to the norm topology and the underlying group operation of the vector space. Although for the majority of applications it is sufficient to know that this operation $+ : X \times X \to X : (x, y) \mapsto x + y$ is continuous, stronger properties of this mapping can be shown. In fact, if $X \times X$ is equipped with the *sum* product metric, then addition becomes a contraction. Examples show that different well-known topological (semi-)groups can be equipped with a natural metric (or gauge of metrics) such that addition is contractive. This *approach group* structure is a canonical generalization of topological groups (or metric groups in the sense of Parthasarathy) and shares some of the important features with the classical concept. For instance, every approach group allows for a natural uniformization.

1. Introduction. For the convenience of the reader we briefly recall some definitions from Lowen and Windels [**6**].

A collection of ideals $(\mathcal{A}(x))_{x\in X}$ in $[0,\infty]$ is called an *approach system* on X if and only if for all $x \in X$ the following conditions are satisfied:

(A1) For all $\varphi \in \mathcal{A}(x) : \varphi(x) = 0$.

(A2) For all $\varphi \in [0,\infty]^X$: (for all $\varepsilon > 0$, for all $N < \infty$: there exists $\varphi_\varepsilon^N \in \mathcal{A}(x)$ such that $\varphi \wedge N \leq \varphi_\varepsilon^N + \varepsilon) \Rightarrow \varphi \in \mathcal{A}(x)$.

(A3) For all $\varphi \in \mathcal{A}(x)$, for all $\varepsilon > 0$, for all $N < \infty$, there exists $(\varphi_z)_{z\in X} \in \prod_{z\in X} \mathcal{A}(z)$ such that for all $y, z \in X$: $\varphi(y) \wedge N \leq \varphi_x(z) + \varphi_z(y) + \varepsilon$.

The pair $(X, (\mathcal{A}(x))_{x\in X})$ is called an *approach space*.

An approach space can also be described by means of a distance function $\delta : X \times 2^X \to [0,\infty]$ or by a limit operator $\lambda : \mathbf{F}(x) \to [0,\infty]^X$,

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