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RECURSIVE SEQUENCES AND FAITHFULLY FLAT EXTENSIONS

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ABSTRACT. For any faithfully flat morphism $A \to B$ of Noetherian normal domains, a power series with coefficients in A which is rational over B, is already rational over A. The proof uses the fact that a sequence is recursive whenever it is recursive over some faithfully flat extension.

0. Introduction. It is well known that a necessary condition for a ring morphism $A \to B$ to be faithfully flat is that *any linear system of equations with coefficients from A which has a solution over B, must already have a solution over A.* In fact, if we strengthen this condition to *any solution over B comes from solutions over A by base change*, then this also becomes a sufficient condition for being faithfully flat. We could paraphrase the necessary condition as follows: *any linear system of equations over A which is solvable over a faithfully flat extension B of A, is already solvable over A.*

In this paper I present another necessary condition of the same flavor. The key definition is that of a (linear) recursive sequence $(x_n)_n$ over a ring A, as a sequence satisfying some fixed linear relation over A among t consecutive terms. I show that if $A \to B$ is faithfully flat and $(x_n)_n$ is a sequence of elements in A satisfying a linear recursion relation with coefficients in B, then it already satisfies such a recursion relation (of the same length) with coefficients in A. As there is a strong connection between recursive sequences and rational power series, I obtain the following corollary. Assume, moreover, that A and B are normal domains; then any power series over A which is rational (meaning that it can be written as a quotient of two polynomials) over B, is already rational over A. Any direct attempt, however, to prove this corollary just using faithfully flatness seems to fail, as far as I can tell.

1. Definition. Let A be a Noetherian ring and let $\mathbf{x} = (x_n)_{n<\omega}$ be a (countable) sequence of elements of A. We say that $\mathbf{x}$ is *recursive over A*

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