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MINIMAL PRESENTATIONS OF FULL SUBSEMIGROUPS OF $\mathbf{N}^2$

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ABSTRACT. We show that the cardinality of a minimal presentation for a two-dimensional full affine subsemigroup of $\mathbf{N}^2$ minimally generated by p elements is $\binom{p-1}{2}$.

A subsemigroup S of $\mathbf{N}^2$ is full if $S = \mathbf{G}(S) \cap \mathbf{N}^2$, where $\mathbf{G}(S)$ denotes the subgroup of $\mathbf{Z}^2$ spanned by S. In this paper we are going to assume that S is a full subsemigroup of $\mathbf{N}^2$ such that $\operatorname{rank}(\mathbf{G}(S)) = 2$. (The case when $\operatorname{rank}(\mathbf{G}(S)) \leq 1$ has no interest, because under this assumption $S = \{(0,0)\}$ or $S \cong \mathbf{N}$.) Note that if $a, b \in S$ and $a - b \in \mathbf{N}^2$, then $a - b \in \mathbf{G}(S) \cap \mathbf{N}^2 = S$. As a consequence, if $M = \{(a_1, b_1), \dots, (a_p, b_p)\}$ is the set of minimal elements of $S \setminus \{0\}$ with respect to the ordering $a \leq b$ if and only if $b - a \in \mathbf{N}^2$, then S is minimally generated by M. Furthermore, we can assume that the elements in M are ordered so that $a_1 < a_2 < \cdots < a_p$ and $b_1 > b_2 > \cdots > b_p$.

We define the map

$$\varphi : \mathbf{N}^p \longrightarrow S$$
$$\varphi(\lambda_1, \dots, \lambda_p) = \sum_{i=1}^{p} \lambda_i (a_i, b_i)$$

and denote its kernel congruence by σ. Clearly, $S \cong \mathbf{N}^p/\sigma$. We say that ρ is a minimal system of generators for σ if ρ generates σ and ρ has minimal cardinality among the generating systems of σ. It can be shown that $\#\rho \geq p - 2$ (see [**5**]).

Given $s \in S \setminus \{0\}$, we define the graph G_s as the graph whose vertices are $V(G_s) = V_s = \{(a_i, b_i) \in M \mid s - (a_i, b_i) \in S\}$ and whose edges are $E(G_s) = E_s = \{[(a_i, b_i), (a_j, b_j)] \mid s - ((a_i, b_i) + (a_j, b_j)) \in S, 1 \leq i, j \leq p\}$.

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