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# EXTENDED RIEMANN ZETA FUNCTIONS

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ABSTRACT. Analogous to recent useful generalizations of the family of gamma functions and beta functions, extensions of the Riemann zeta function are presented, for which the usual properties and representations are naturally and simply extended. In analogy to these extensions, the extended Hurwitz functions are introduced. Hurwitz-type formulae are also proved.

**1. Introduction.** The zeta function, though originally introduced by Euler, was independently used by Riemann to attack a problem in the theory of prime numbers [**4**], [**8**], [**11**], [**13**]. It was known that prime numbers become progressively sparser for large values but no explicit expression explaining how they become so was available until the time of Legendre and Gauss. Writing the number of primes less than or equal to $n$ as

$$\pi(n) = \sum_{p \le n} 1, \tag{1.1}$$

where the summation extends only over the primes. At age 15 Gauss, in 1792, conjectured that as $n \to \infty$,

$$\pi(n) \sim n/\log n. \tag{1.2}$$

In an attempt to prove the conjecture, Riemann used the zeta function extended to complex variables. (It may be mentioned in passing that though the Riemann proof was incomplete, there is now available an

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