ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 4, Winter 2001

WHEN THE FAMILY OF FUNCTIONS VANISHING AT INFINITY IS AN IDEAL OF $C(X)$

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ABSTRACT. We prove that $C_\infty(X)$ is an ideal in $C(X)$ if and only if every open locally compact subset of X is bounded. In particular, if X is a locally compact Hausdorff space, $C_\infty(X)$ is an ideal of $C(X)$ if and only if X is a pseudocompact space. It is shown that the existence of some special functions in $C_\infty(X)$ causes $C_\infty(X)$ not to be an ideal of $C(X)$. Finally we will characterize the spaces X for which $C_\infty(X)$ and $C_K(X)$, or $C_\psi(X)$, coincide.

Introduction. Throughout this paper X stands for a completely regular Hausdorff space and $C(X)(C^*(X))$ for the ring of all (bounded) continuous real valued functions on X. In [**1**], Azarpanah considered essential ideals in $C(X)$ and characterized those X for which the ideal $C_K(X)$ of all functions in $C(X)$ with compact support is an essential ideal in $C(X)$. He considered also the subset $C_\infty(X)$ of all those functions in $C(X)$ which vanish at infinity. It gives an impression there that $C_\infty(X)$ might always be an ideal of $C(X)$. This, however, is not always true, e.g., $X = \mathbf{R}$.

We prove that $C_\infty(X)$ will be an ideal of $C(X)$ if and only if every open locally compact subset of X is bounded. In particular, for a locally compact Hausdorff space X, $C_\infty(X)$ is an ideal in $C(X)$ if and only if X is a pseudocompact space. We note that $Y \subseteq X$ is said to be bounded if for every $f \in C(X)$, $f(Y)$ is a bounded set in $\mathbf{R}$. We will show that the existence of a function $f \in C_\infty(X) \setminus C_K(X)$ whose zero-set $Z(f)$ is an open set, causes $C_\infty(X)$ not to be an ideal of $C(X)$. We also observe that the existence of a function h in $C_\infty(X)$ with $Z(h)$ a Lindelöf and bounded set causes $C_\infty(X)$ not to be an ideal of $C(X)$, unless X is a compact space.

1991 *Mathematics Subject Classification.* 54C40.
The research of the first author is partially supported by the Institute for Studies in Theoretical Physics and Mathematics (IPM), Tehran.
Received by the editors on May 9, 2000, and in revised form on November 6, 2000.