ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 3, Fall 2001

ONE-SIDED TAUBERIAN THEOREMS FOR DIRICHLET SERIES METHODS OF SUMMABILITY

DAVID BORWEIN, WERNER KRATZ AND ULRICH STADTMÜLLER

ABSTRACT. We extend recently established two-sided or O-Tauberian results concerning the summability method $D_{\lambda,a}$ based on the Dirichlet series $\sum a_n e^{-\lambda_n x}$ to one-sided Tauberian results. More precisely, we formulate one-sided Tauberian conditions, under which $D_{\lambda,a}$-summability implies convergence. Our theorems contain various known results on power series methods of summability and, in the so-called high index case we even obtain a new result for such methods. Our method of proof uses asymptotic properties of the Dirichlet series subject to the assumption that a_n and λ_n can be interpolated by smooth functions. In addition we develop refined Vijayaraghavan-type results which enable us to infer the boundedness of sequences from the boundedness of their $D_{\lambda,a}$-means and the one-sided Tauberian conditions.

1. Introduction and main results. Suppose throughout that $\{\lambda_n\}$ is an unbounded and strictly increasing sequence of positive numbers, that $\{a_n\}$ is a sequence of nonnegative numbers, and that the Dirichlet series

$$a(x) := \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}$$

has abscissa of convergence $\sigma \in [-\infty, \infty)$. Let $\{s_n\}$ be a sequence of *real* numbers. The Dirichlet series summability method $D_{\lambda,a}$ is defined as follows:

$$s_n \to s\,(D_{\lambda,a}) \quad \big\{\text{or } s_n = O(1)(D_{\lambda,a})\big\}$$

$$\text{if} \quad \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \quad \text{is convergent for} \quad x > \sigma, \quad \text{and}$$

$$\sigma(x) := \frac{1}{a(x)} \sum_{n=1}^{\infty} a_n s_n e^{-\lambda_n x} \to s \quad \big\{\text{or } \sigma(x) = O(1)\big\} \quad \text{as } x \to \sigma+$$

Research supported in part by the Natural Sciences and Engineering Research Council of Canada.
1991 AMS *Mathematics Subject Classification.* 40G10, 40E05.
Key words and phrases. Tauberian, Dirichlet series methods.
Received by the editors on April 20, 1999, and in revised form on May 22, 2000.