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GEOMETRIC THEOREMS AND PROBLEMS FOR HARMONIC MEASURE

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ABSTRACT. We review some results and open problems for harmonic measure. Their common element is their simple geometric character. Such classical results are the projection estimates of Beurling, Nevanlinna and Hall. Recent projection theorems are due to Marshall, Sundberg, Solynin and others. One of the methods of the proofs of these theorems is the method of extremal metrics and quadratic differentials. The classical symmetrization result for harmonic measure (originally proven by the star-function method of Baernstein) can now be proven by the polarization technique. There remain, however, several open problems. For most of these problems there are reasonable conjectures. The paper discusses the above results, problems and methods. It contains several new open problems.

1. Introduction. Harmonic measure is one of the most important conformal invariants. It is a standard tool in complex analysis which provides a connection between potential theory and geometric function theory. The name *harmonic measure* was introduced by Nevanlinna [**56**] in 1934 but methods related to harmonic measure had been used much earlier by Lindelöf, Carleman, Ostrowski and F. and R. Nevanlinna.

Let D be a domain in the complex plane $\mathbf{C}$, and let E be a Borel set on the boundary ∂D of D. The harmonic measure of E at $z \in D$ relative to D is the Perron solution $u(z)$ of the Dirichlet problem in D with boundary values 1 on E and 0 on $\partial D \setminus E$. More precisely, let $\chi_E = 1$ on E and $\chi_E = 0$ on $\partial D \setminus E$. Then

$$u(z) = \sup\{v(z) : v \text{ subharmonic in } D \text{ and } \limsup_{w\to\zeta} v(w) \le \chi_E(\zeta) \text{ for } \zeta \in \partial D\}.$$

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