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CONFIGURATIONS OF CYCLES AND THE APOLLONIUS PROBLEM

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ABSTRACT. Given $n+1$ spheres and planes of dimension $n-1$ in $\mathbf{R}^n$, the Apollonius problem is to find a common tangent sphere or plane, and the generalized Apollonius problem is to find a sphere or plane intersecting them under prescribed angles. In Lie geometry, an Apollonius problem is given by an $(n+1)$-frame of points on the Lie quadric $\Omega \subset \mathbf{P}^n + 2$. The solutions are described as the intersections of the projective line determined by the orthogonal complement to this frame with respect to the Lie product in $\mathcal{R}^{n+3}$ and the quadric. Two special points span this line, and the connection between the position of these two points and the existence and geometric properties of the solutions of the Apollonius problem are described.

1. Introduction. In Lie geometry, planes and spheres of dimension $n-1$ in $\mathbf{R}^n$ are described as points on a quadric in the projective space $\mathbf{P}^{n+2}$ and the angle of intersection is expressed in terms of the Lie product in $\mathbf{R}^{n+3}$. Lie geometry is a natural environment for describing certain geometric constructions, for example the Apollonius construction where we look for an object which is tangent to a given set of objects. The objects involved are either planes, spheres or points, where points count as spheres with radius zero and both together will be called *geometric cycles*. Tangency and intersection of geometric cycles correspond to algebraic relations between the corresponding points of the projective space. This means that to find a solution to a geometric construction it suffices to solve a system of algebraic equations.

In this paper we discuss how geometric properties of the solution set of certain constructions in $\mathbf{R}^n$ given by $n+1$ cycles can be reconstructed from the position of their corresponding points in $\mathbf{P}^{n+2}$.

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