ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 2, Summer 2001

CONSTRUCTION OF WEIGHT TWO EIGENFORMS VIA THE GENERALIZED DEDEKIND ETA FUNCTION

DONALD L. VESTAL, JR.

ABSTRACT. The generalized Dedekind eta function has been used in various ways to construct modular functions of different weights. In this paper we give a way to construct modular forms of weight two for the modular groups $\Gamma_0(N)$ which, in some cases, turn out to be Hecke eigenforms (though never cusp forms).

1. The generalized Dedekind eta function. Let $\mathfrak{h}$ denote the upper half plane (so $\mathfrak{h} = \{\tau \mid \operatorname{Im}\tau > 0\}$), and let $P_2(x) = \{x\}^2 - \{x\} + (1/6)$ denote the second Bernoulli polynomial, defined on the fractional part of x, $\{x\} = x - \lfloor x \rfloor$. For integers g and δ, with $\delta > 0$, we define the generalized Dedekind eta function as

$$(1) \quad \eta_{\delta,g}(\tau) = e^{\pi i \delta P_2(g/\delta)\tau} \prod_{\substack{m \equiv g \pmod{\delta} \\ m>0}} (1-q^m) \prod_{\substack{m \equiv -g \pmod{\delta} \\ m>0}} (1-q^m)$$

where $\tau \in \mathfrak{h}$ and $q = e^{2\pi i \tau}$. These functions are a variation of the eta functions defined by Schoeneberg in [**5**] and can be used to create modular functions in various ways (see [**4**] and [**6**]). For example, from [**6**], we have

Theorem. *Let N be a positive integer, and let*

$$f(\tau) = \prod_{\substack{\delta \mid N \\ 0 \le g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau),$$

where $r_{\delta,g} \in Z$ and $r_{\delta,ag} = r_{\delta,g}$ for all a relatively prime to N. Set

Received by the editors on January 14, 2000, and in revised form on February 15, 2000.