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GROWTH, DISTORTION AND COEFFICIENT BOUNDS FOR PLANE HARMONIC MAPPINGS CONVEX IN ONE DIRECTION

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ABSTRACT. In this paper we examine normalized harmonic functions convex in the direction of either the real or the imaginary axis. In this setting we find bounds for $|f_z(z)|$, $|f_{\bar{z}}(z)|$ and $|f(z)|$, as well as coefficient bounds on the series expansion of functions convex in the direction of the real axis. For the functions convex in the direction of the real axis, we provide the extremal functions for $|f_z(z)|$ and $|f_{\bar{z}}(z)|$.

Many important questions in the study of classes of functions relate to bounds on the modulus of the function (growth) or the modulus of the derivative (distortion). In this paper we examine both of these questions, as well as coefficient bounds, for two classes of complex-valued harmonic functions of one complex variable.

Any harmonic function in the open unit disk $\mathbf{D}$ can be written as a sum of an analytic and anti-analytic function, $f(z) = h(z) + \overline{g(z)}$. The Jacobian of the mapping f, denoted $J_f(z)$, can be computed by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. If the function f is locally univalent, as will be true of all functions in this work, then $J_f(z) \neq 0$ for $z \in \mathbf{D}$. For convenience, we will only examine *sense-preserving* functions, that is, functions for which $J_f(z) > 0$. If f has $J_f(z) < 0$, then $\bar{f}$ is sense-preserving. The *analytic dilatation* of a harmonic function is the quantity $\omega(z) = (g'(z)/h'(z))$. Note that if f is locally univalent and sense-preserving, $|\omega(z)| < 1$. The class of functions f in S_H is defined by

$$S_H = \{f = h + \bar{g} : f \text{ is univalent in } \mathbf{D} \\ \text{and satisfies } f(0) = 0, h'(0) = 1\},$$

and the compact normal family S_H^0 is defined by

$$S_H^0 = \{f = h + \bar{g} : f \in S_H \text{ and } g'(0) = 0\}.$$

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