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ON JACOBI'S THEOREM IN HAMILTON-JACOBI THEORY

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ABSTRACT. Jacobi's theorem states that a complete integral of the Hamilton partial differential equation for a given Hamiltonian determines in a simple way all the trajectories of the Hamiltonian flow. It is usually proved by appealing to the theory of canonical transformations. Our approach consists in noting a fact which is actually at the center of the existing proofs, whose proof is just a simple differentiation, and which doesn't seem to have been noticed so far: Given a one-parameter family of solutions of the Hamilton-Jacobi differential equation, its partial derivative with respect to the parameter is an integral for the corresponding field curves. Jacobi's theorem is an immediate consequence of this, without any further computation.

We recall: A Hamiltonian is a function H of $2n+1$ real variables q_i, p_i, t with $i = 1, \dots, n$, defined in some open set D in $\mathbf{R}^{2n+1}$. (We shall use notations like q for the point $(q_1, q_2, \dots, q_n)$ and H_q for the sequence of partial derivatives $(H_{q_1}, \dots, H_{q_n})$).

A trajectory or extremal (for H) is a curve $(q(t), p(t), t)$ in $\mathbf{R}^{2n+1}$, defined on some t-interval, that lies in D and satisfies the canonical or Hamilton equations

$$dq/dt = H_p(q(t), p(t), t), \qquad dp/dt = -H_q(q(t), p(t), t). \tag{0.1}$$

There is the associated Lagrangian, a function L of $2n+1$ variables $q_1, q_2, \dots, q_n,\ \dot{q}_1, \dot{q}_2, \dots, \dot{q}_n, t$. First one maps the original domain D to a domain D' in $(q, \dot{q}, t)$-space by the map (assumed to be a diffeomorphism) given by the identity on q and t and

$$\dot{q} = H_p(q, p, t). \tag{0.2}$$

Then one defines L by the relation

$$L(q, \dot{q}, t) + H(q, p, t) = \sum p_i \cdot \dot{q}_i. \tag{0.3}$$

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