ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 31, Number 1, Spring 2001

UNIVALENCE AND CONVEXITY PROPERTIES FOR GAUSSIAN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. Let $\mathcal{A} = \{f : \Delta \to \mathbf{C} \,|\, f(z) = z + \sum_{n=2}^{\infty} A_n z^n\}$. We study sufficient/necessary conditions, in terms of the coefficients A_n, for a function $f \in \mathcal{A}$ to be member of well-known subclasses of the class $\mathcal{S}$ of univalent functions. Examples of these subclasses include starlike, convex, close-to-convex functions. In particular, functions of the form $z_2F_1(a, b; c; z)$ are considered, where ${}_2F_1(a, b; c; z)$ is the hypergeometric function.

1. Introduction. The class of normalized analytic functions

$$\mathcal{A} = \left\{ f : \Delta \to \mathbf{C} \,\middle|\, f(z) = z + \sum_{n=2}^{\infty} A_n z^n \right\} \tag{1.1}$$

has been studied extensively, together with its subclass of univalent (Schlicht) functions

$$\mathcal{S} = \{f \in \mathcal{A} \,|\, f \text{ is one-to-one in } \Delta\}, \tag{1.2}$$

where Δ is the unit disc. Along with the classes $\mathcal{A}$ and $\mathcal{S}$ several subclasses of $\mathcal{S}$ have been widely studied. Two such subclasses are analytically characterized by

$$\mathcal{C}(\beta) = \left\{ f \in \mathcal{A} \,\middle|\, \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta,\ z \in \Delta \right\}, \quad \beta < 1, \tag{1.3}$$

and

$$\mathcal{S}^*(\beta) = \left\{ f \in \mathcal{A} \,\middle|\, \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \Delta \right\}, \quad \beta < 1. \tag{1.4}$$

Received by the editors on June 23, 1998.
1991 AMS *Mathematics Subject Classification.* Primary 30C45, Secondary 33A30.
Key words and phrases. Gaussian hypergeometric functions, univalent, convex and starlike functions.