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A WEAK HARNACK INEQUALITY FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT. We prove a priori estimates for nonnegative supersolutions of fractional differential equations of the form $\partial_t^\alpha(u-u_0)+\mu\, u = f$, $u(0)=u_0$, with $\alpha \in (0,1)$. As a main result, we establish for such functions a weak Harnack inequality with critical exponent $1/(1-\alpha)$, which is shown to be optimal. In addition, we obtain an L_p-estimate of Moser type and show that positive supersolutions satisfy certain log-estimates; the latter plays a crucial role in connection with an abstract lemma of Bombieri and Giusti, which is an extremely useful tool to prove Harnack-type estimates for a wide class of elliptic and parabolic problems. Therefore, the results obtained are also of preliminary character with regard to a corresponding theory for fractional evolution equations of the form $\partial_t^\alpha(u-u_0)-Lu=f$, where L stands for a uniformly elliptic operator of second order.

1. Introduction. Harnack inequalities have proved to be a powerful tool in the theory of linear and nonlinear partial differential equations. The classical parabolic Harnack inequality is due to Hadamard [**10**] and Pini [**22**]. A seminal contribution in this field was then made by Moser [**17, 18**], who established a Harnack inequality for weak solutions of second order elliptic differential equations in divergence form with merely bounded measurable coefficients. By means of this result he was able to give a new proof of the well-known De Giorgi-Nash theorem ([**6, 21**]) on the Hölder continuity of weak solutions of such equations. Among the most important works on Harnack inequalities are further Moser [**19**] and Krylov-Safonov [**12**], which deal with parabolic differential equations in divergence, respectively non-divergence form. In all these papers the operators under study are local operators, that is, differential operators.

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