JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 6, Number 2, Spring 1994

PALEY-WIENER THEOREM AND THE FACTORIZATION

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ABSTRACT. In this note we shall generalize the Paley-Wiener theorem to self-adjoint operators similar to $(-id/dx)$. By using a simple factorization result, it is shown that the Paley-Wiener theorem holds if $\Gamma'(\lambda)$ is analytic, where $\Gamma(\lambda)$ is the spectral function.

1. Introduction. Recall that with each self-adjoint operator is associated a transform or a unitary operator by which the self-adjoint operator is equivalent to a multiplication by the independent variable. For instance, $-id/dx$ is self-adjoint in the Hilbert space L^2_{dx} and $\mathcal{F}(f)(\lambda) = \int_{\mathbf{R}} f(x)e^{i\lambda x}\,dx$ defines a unitary operator called the Fourier transform

$$L^2_{dx} \xrightarrow{\mathcal{F}} L^2_{d\lambda/2\pi}.$$

One of the most interesting features of the Fourier transform is the Paley-Wiener theorem: Let $F(\lambda)$ be an entire function

$$\left.\begin{array}{c} |F(\lambda)| < Me^{a|\lambda|} \\ F(\lambda) \in L^2_{d\lambda} \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} F(\lambda) = \displaystyle\int_{-a}^{a} f(x)e^{i\lambda x}\,dx \\ f(\lambda) \in L^2_{dx} \end{array}\right.$$

$e^{i\lambda x}$ are clearly the eigenfunctionals of the operator $-id/dx$. Our question is: Characterize self-adjoint operators in $L^2_{dM(x)}$ such that if $e^{i\lambda x}$ is replaced by its eigenfunctionals, then does a similar Paley-Wiener theorem hold? For the sake of simplicity, it is sufficient to consider self-adjoint operators with a simple spectrum, σ say. Let L be a self-adjoint operator acting in the separable Hilbert space $L^2_{dM(x)}$, and let $y(x,\lambda)$ be its eigenfunctionals, i.e., $Ly(x,\lambda) = \lambda y(x,\lambda)$ in the weak sense, see [**3**]. This gives rise to the y-transform, F_y

$$\forall\, f \in L^2_{dM(x)} \quad F_y(f)(\lambda) \equiv \int f(x)y(x,\lambda)\,dM(x).$$

Received by the editors on June 28, 1993 and in revised form on July 11, 1994.
AMS *Mathematics Subject Classification.* 46, 47.