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NONLINEAR EQUATIONS INVOLVING NONPOSITIVE DEFINITE LINEAR OPERATORS VIA VARIATIONAL METHODS

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ABSTRACT. In this paper we establish, via variational methods, some existence results for nonlinear equations of the type $u = K\mathbf{f}(u)$, where $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$ is linear and $\mathbf{f} : L^p(\Omega) \to L^q(\Omega)$ is a superposition operator with $p_0 > p > 2$, $p^{-1} + q^{-1} = 1$ and $p_0^{-1} + q_0^{-1} = 1$. Then we apply these results to study a Hammerstein equation and a nonresonant nonlinear Fredholm integral equation. Our approach allows us to deal with nonpositive definite kernels. This is a novelty for the application of variational methods when coercivity fails to hold.

1. Preliminaries and basic definitions. Throughout this paper p_0, q_0 are two real numbers with $p_0 > 2$ and $1/p_0 + 1/q_0 = 1$, $\Omega \subseteq \mathbf{R}^N$ is a bounded Lebesgue measurable set, $K : L^{q_0}(\Omega) \to L^{p_0}(\Omega)$ is a completely continuous linear operator and $f : \Omega \times \mathbf{R} \to \mathbf{R}$ is a Carathéodory function. Consider p and q, with $2 < p < p_0$ and $1/p + 1/q = 1$; we suppose that $\mathbf{f}(u) \in L^q(\Omega)$, for every $u \in L^p(\Omega)$, where $\mathbf{f}(u) = f(\cdot, u(\cdot))$ denotes the superposition operator associated to f. Moreover, $\|\cdot\|_m$ will denote the usual norm in $L^m(\Omega)$ for $m \geq 1$. We are interested in finding solutions in $L^p(\Omega)$ to the following equation

$$u = K\mathbf{f}(u). \tag{1.1}$$

Equation (1.1) has been studied by several authors. One of the reasons which leads us to consider equation (1.1) concerns the fact that various boundary value problems for differential equations can be reduced to an integral equation like (1.1). For example, we refer to the case in which the operator K is defined by

$$K(u)(\cdot) = \int_\Omega k(\cdot, y) u(y)\, dy \tag{1.2}$$

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