JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 16, Number 4, Winter 2004

AN ABSTRACT GRONWALL LEMMA AND APPLICATIONS TO GLOBAL EXISTENCE RESULTS FOR FUNCTIONAL DIFFERENTIAL AND INTEGRAL EQUATIONS OF FRACTIONAL ORDER

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1. Introduction. The aim of this paper is two-fold. On the one hand, we prove an abstract generalization of a Gronwall lemma which gives *a priori* estimates for various (functional) differential and integral equations, of Volterra type, under a linear growth condition on the nonlinearity. We believe that this result is of independent interest and discuss it in a rather general setting. On the other hand, we apply a simple special case of this abstract result to obtain the existence of *global* solutions of the functional differential equation of fractional type

(1)
$$D^\alpha x(t) = f\big(t, x(t-c_1), \dots, x(t-c_n), D^{\alpha_1}x(t-a_1), \dots, D^{\alpha_k}x(t-a_k), I^{\beta_1}x(t-b_1), \dots, I^{\beta_m}x(t-b_m)\big)$$

under a linear growth condition on f. Here, $a_j, b_j, c_j \geq 0$, and $\alpha > \alpha_j > 0$ denote the, not necessarily integer, order of the corresponding (either Riemann-Liouville or Caputo) differential operators while $\beta_j > 0$ denote the, not necessarily integer, order of the (Abel) integral operators. We also consider inclusion problems of the type (1).

For $n = m = 0$, i.e., if the righthand side depends only on $(t, D^{\alpha_1}x(t-a_1), \dots, D^{\alpha_k}x(t-a_k))$, equation (1) has provoked some interest in the literature [**1, 2, 7, 9–12, 14, 25**]. In comparison with the existence results in these references, our assumptions are more natural. In contrast to these references, we only require that f has a linear growth and need not assume that this linear growth is sufficiently small. Of course, we can do this only because we obtain the required *a priori* estimate for the solution by means of our Gronwall lemma. We also drop the requirement that f is real-valued and consider the general

This paper was written in the framework of a Heisenberg Fellowship of the second author (Az. VA 206/1-1). Financial support by the DFG is gratefully acknowledged.
Received by the editors on June 12, 2004.