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ON INTEGRAL EQUATIONS ARISING IN THE FIRST-PASSAGE PROBLEM FOR BROWNIAN MOTION

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ABSTRACT. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g:(0,\infty)\to\mathbf{R}$ be a continuous function satisfying $g(0+)\geq 0$, let

$$\tau = \inf\{t>0 \mid B_t \geq g(t)\}$$

be the first-passage time of B over g, and let F denote the distribution function of τ. Then the following system of integral equations is satisfied:

$$t^{n/2} H_n\left(\frac{g(t)}{\sqrt{t}}\right) = \int_0^t (t-s)^{n/2} H_n\left(\frac{g(t)-g(s)}{\sqrt{t-s}}\right) F(ds)$$

for $t>0$ and $n=-1,0,1\ldots$, where $H_n(x)=\int_x^\infty H_{n-1}(z)\,dz$ for $n\geq 0$ and $H_{-1}(x)=\varphi(x)=(1/\sqrt{2\pi})e^{-x^2/2}$ is the standard normal density. These equations are derived from a single 'master equation' which may be viewed as a Chapman-Kolmogorov equation of Volterra type. The initial idea in the derivation of the master equation goes back to Schrödinger [**23**].

1. Introduction. Let $(B_t)_{t\geq 0}$ be a standard Brownian motion started at zero, let $g:(0,\infty)\to\mathbf{R}$ be a continuous function satisfying $g(0+)\geq 0$, let

$$\tau = \inf\{t>0 \mid B_t \geq g(t)\} \tag{1.1}$$

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