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AN INTEGRAL OPERATOR SOLUTION TO THE MATRIX TODA EQUATIONS

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ABSTRACT. In previous work the author found solutions to the Toda equations that were expressed in terms of determinants of integral operators. Here it is observed that a simple variant yields solutions to the matrix Toda equations. As an application another derivation is given of a differential equation of Sato, Miwa and Jimbo for a particular Fredholm determinant.

During the last 20 years, beginning with [**2**], many connections have been established between determinants of integral operators and solutions of differential equations. A result proved in [**2**] can be shown to be equivalent to one concerning the integral operator K on $L^2(\mathbf{R}^+)$ with kernel

$$\frac{e^{-t(u+u^{-1}+v+v^{-1})/4}}{u+v}.$$

It is that the function $\tau := \log\det(I-\lambda^2K^2)$ has the representation

$$\tau = -\frac{1}{2}\int_t^\infty s\left(\left(\frac{d\varphi}{ds}\right)^2 - \sinh^2\varphi\right)ds, \tag{1}$$

where $\varphi = \varphi(t;\lambda)$ satisfies the differential equation

$$\frac{d^2\varphi}{dt^2} + \frac{1}{t}\frac{d\varphi}{dt} = \frac{1}{2}\sinh 2\varphi \tag{2}$$

with boundary condition

$$\varphi(t;\lambda) \sim 2\lambda K_0(t) \quad \text{as } t \longrightarrow \infty.$$

(Here K_0 is the usual modified Bessel function.) The differential equation for φ, the cylindrical sinh-Gordon equation, is reducible to a special case of the Painlevé III equation. The result of [**2**] was the

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