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RADON TRANSFORM OVER CONES AND RELATED DECONVOLUTION PROBLEMS

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ABSTRACT. We introduce a new kind of radon transform, consisting in integrating a function (to be recovered) over a special family of cones. It is in fact a formal generalization of the "Coded-aperture gammagraphy" imaging method, encountered in medicine and astronomy. We show that it is a natural geometric operation, but which does not have the fine properties of similar integral transform. Nevertheless, several inverse problems (like classical radon transform, deconvolution) are related to it, and also new kinds on integral transforms : essentially the "Quasi-convolution". After this study, where we show that the problem is severely ill-posed (essentially because of insufficient data), an inversion is performed in the case of complete data.

1. Introduction.

1.1 Notations and tools. A point $x \in \mathbf{R}^n$ is written $x = (x', x_n)$. The Euclidean scalar product of x and y is $x.y$, and the associated norm of x is $|x|$. We denote closed balls $\mathbf{B}(a, r) = \{x \in \mathbf{R}^n,\ |x - a| \leq r\}$, spheres $\mathbf{S}(\mathbf{a}, \mathbf{r}) = \{\mathbf{x} \in \mathbf{R}^\mathbf{n},\ |\mathbf{x} - \mathbf{a}| = \mathbf{r}\}$. Let $\mathbf{B}(0,1) = \mathbf{B}$, $\mathbf{S}(0,1) = \mathbf{S}$, and call $\mathbf{S}_+$, the half-unit sphere of $\mathbf{R}^n$ for positive x_n.

We introduce $\mathbf{\Pi}(\lambda) = \{x \in \mathbf{R}^n,\ x_n = \lambda\}$, and let $\mathbf{\Pi}(0) = \mathbf{\Pi}$, the hyperplane delimiting the two open half-spaces $\mathbf{R}^n_+ = \{x \in \mathbf{R}^n,\ x_n > 0\}$ and $\mathbf{R}^n_- = \{x \in \mathbf{R}^n,\ x_n < 0\}$. The set $\mathbf{\Pi}$ will be called the "plane of the code," and the previous half-spaces will respectively be the "region of the source" and the "region of the detector."

The characteristic function of a compact set $\mathbf{K}$ of $\mathbf{R}^n$ is denoted

$$\chi_{\mathbf{K}}(x) = \begin{cases} 1 & \text{if } x \in \mathbf{K}, \\ 0 & \text{if } x \notin \mathbf{K}. \end{cases}$$

For $x \in \mathbf{R}^n$ and $\lambda \in \mathbf{R}$, we write $f_\lambda(x) = \lambda^n f(\lambda x)$.

1.1.1 *The Fourier transform and the convolution.* In the context of the spaces $L^p = \{f : \mathbf{R}^n \mapsto \mathbf{R}\ ;\ \left(\int_{\mathbf{R}^n} |f(x)|^p\, dx\right)^{1/p} = \|f\|_p < +\infty\}$,

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