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MOSER'S MATHEMAGICAL WORK ON THE EQUATION $1^k + 2^k + \cdots + (m-1)^k = m^k$

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In memory of Alf van der Poorten (1942–2010)

ABSTRACT. If the equation of the title has an integer solution with $k \geq 2$, then $m > 10^{10^6}$. Leo Moser showed this in 1953 by amazingly elementary methods. With the hindsight of more than 50 years, his proof can be somewhat simplified. We give a further proof showing that Moser's result can be derived from a von Staudt-Clausen type theorem. Based on more recent developments concerning this equation, we derive a new result using the divisibility properties of numbers in the sequence $\{2^{2e+1}+1\}_{e=0}^{\infty}$. In the final section we show that certain Erdős-Moser type equations arising in a recent paper of Kellner can be solved completely.

1. Introduction. In this paper we are interested in non-trivial solutions, that is, solutions with $k \geq 2$, of the equation

$$1^k + 2^k + \cdots + (m-2)^k + (m-1)^k = m^k. \tag{1}$$

The conjecture that such solutions do not exist was formulated around 1950 by Paul Erdős in a letter to Leo Moser. For $k = 1$, one has the solution $1 + 2 = 3$ (and no further solutions). From now on, we will assume that $k \geq 2$. Moser [**29**] established the following theorem in 1953.

Theorem 1 [**29**]. *If (m, k) is a solution of* (1), *then $m > 10^{10^6}$.*

His result has since been improved. Butske et al. [**6**] have shown by computing rather than estimating certain quantities in Moser's original proof that $m > 1.485 \cdot 10^{9\,321\,155}$. By proceeding along these lines this bound cannot be substantially improved. Butske et al. [**6**, page 411]

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