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ISOSPECTRAL MEASURES

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ABSTRACT. In recent papers a number of authors have considered Borel probability measures μ in $\mathbf{R}^d$ such that the Hilbert space $L^2(\mu)$ has a Fourier basis (orthogonal) of complex exponentials. If μ satisfies this property, the set of frequencies in this set is called a spectrum for μ. Here we fix a spectrum, say Γ, and we study the possibilities for measures μ having Γ as spectrum.

1. Introduction. We consider a spectral analysis of families of singular measures, introducing pairs (μ, Γ) where μ is a measure, and where Γ is a set which serves as spectrum for μ; see the definition below. We refer to these as spectral pairs. While the measures arising in this way have a special flavor, they are nonetheless useful in the analysis of models arising in a host of different areas.

We are motivated in part by a renewed interest in families of singular measures, driven in turn both by applications, and by current problems in spectral theory and geometric measure theory. The applications include Schroedinger operators from physics, especially their scattering theory [**1, 4, 14**]. In these problems, it is helpful to have at hand concrete model-examples involving measures amenable to direct computations. In stochastic processes and stochastic integration, key tools depend on underlying spectral densities. For problems involving fluctuations and chaotic dynamics, the measures are often singular and model-measures are helpful, see e.g., [**2, 3, 5, 15, 24, 28**]. In determining the nature of orbits in ergodic theory, the first question is often "what is the spectral type?" The measures in these applications are typically not compactly supported. Nonetheless, there is a procedure from geometric measure theory which produces compactly supported measures, see [**17**], and much of the earlier literature has focused on measures of compact support. Our results below show that non-compactly supported measures arise in every spectral pair.

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