ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 43, Number 5, 2013

SHARP INEQUALITIES INVOLVING THE POWER MEAN AND COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

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ABSTRACT. In this paper, we prove that $M_p(\mathcal{K}(r),\mathcal{K}(r')) \ge \mathcal{K}(\sqrt{2}/2)$ and $M_q(\mathcal{K}(r),\mathcal{K}(r')) \le \mathcal{K}(\sqrt{2}/2)$ for all $r \in (0,1)$ if and only if $p \ge 1 - 4[\mathcal{K}(\sqrt{2}/2)]^4/\pi^2 = -3.789\ldots$ and $q \le (\log 2)/[\log(\pi/2) - \log\mathcal{K}(\sqrt{2}/2)] = -4.180\ldots$, where $\mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta$ is the complete elliptic integral of the first kind, $r' = \sqrt{1-r^2}$, and $M_p(x,y)$ is the power mean of order p of two positive numbers x and y.

1. Introduction. Throughout this paper, we denote $r' = \sqrt{1-r^2}$ for $0 < r < 1$. The well-known complete elliptic integrals of the first and second kinds [**13, 15**] are defined by

$$\begin{cases} \mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta, \\ \mathcal{K}' = \mathcal{K}'(r) = \mathcal{K}(r'), \\ \mathcal{K}(0) = \pi/2, \quad \mathcal{K}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E} = \mathcal{E}(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{1/2}d\theta, \\ \mathcal{E}' = \mathcal{E}'(r) = \mathcal{E}(r'), \\ \mathcal{E}(0) = \pi/2, \quad \mathcal{E}(1) = 1, \end{cases}$$

respectively.

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory,

2010 AMS *Mathematics subject classification.* Primary 33E05, 26E60.
Keywords and phrases. Complete elliptic integrals, power mean, inequality.
The research of this paper was supported by the NSF of China under grants 11071069 and 11171307, and the NSF of Zhejiang Province under grant LY13A010004.
Received by the editors on January 6, 2011, and in revised form on March 31, 2011.
DOI:10.1216/RMJ-2013-43-5-1489