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A CONSTRUCTIVE PROOF OF THE BISHOP-PHELPS-BOLLOBÁS THEOREM FOR THE SPACE $C(K)$

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ABSTRACT. In 1961 Bishop and Phelps proved that every real Banach space is subreflexive; in other words, every continuous linear functional can be approximated by a norm-attaining functional of the same norm. In 1970, Bollobás refined the result to show that, if f is a norm one functional that "almost" attains its norm at a point x, then f can be approximated by a norm attaining norm one functional that attains its norm at a point close to x. The original proof of the Bishop-Phelps result is an existence argument. We give a constructive proof of the Bishop-Phelps-Bollobás theorem for the real space $C(K)$.

1. Introduction. In 1961, Bishop and Phelps [**1**] proved their well-known theorem that every real Banach space is subreflexive; in other words, the set of support functionals for a closed, bounded, convex subset S of a real Banach space X is norm dense in X^*. In 1970, Bollobás [**2**] showed the following refinement of the Bishop-Phelps result:

Theorem 1.1. *Denote by S and S' the unit spheres in a real Banach space E and its dual space E', respectively. Suppose $x \in S$, $f \in S'$ and $|f(x) - 1| \le \varepsilon^2/2$ $(0 < \varepsilon < 1/2)$. Then there exist $y \in S$ and $g \in S'$ such that $g(y) = 1$, $\|f - g\| \le \varepsilon$ and $\|x - y\| < \varepsilon + \varepsilon^2$.*

In other words, if f is a norm one functional that "almost" attains its norm at a point x, then f can be approximated by a norm attaining norm one functional that attains its norm at a point close to x.

The result that follows relates to Bollobás's formulation of the theorem. We begin by giving a constructive proof of the Bishop-Phelps-Bollobás theorem in the real space $C(K)$.

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